

AUTOMORPHISMS OF MODULI SPACES OF VECTOR BUNDLES OVER A CURVE

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ABSTRACT. Let X be an irreducible smooth complex projective curve of genus $g \geq 4$. Let $M(r, \Lambda)$ be the moduli space of stable vector bundles $E \rightarrow X$ of rank r and fixed determinant Λ . We show that the automorphism group of $M(r, \Lambda)$ is generated by automorphisms of the curve X , tensorization with suitable line bundles, and, if r divides $2\deg(\Lambda)$, also dualization of vector bundles.

1. INTRODUCTION

Let X be a smooth complex projective curve of genus g , with $g \geq 4$. Fix a line bundle Λ on X of degree d . Let $M(r, \Lambda)$ be the moduli space of stable vector bundles $E \rightarrow X$ of rank r and determinant $\det(E) \cong \Lambda$.

There are two obvious ways of producing automorphisms of $M(r, \Lambda)$:

- (1) Let $\sigma : X \rightarrow X$ be an automorphism, and let L be a line bundle with $L^r \otimes \sigma^* \Lambda \cong \Lambda$. Send E to $L \otimes \sigma^* E$.
- (2) Let $\sigma : X \rightarrow X$ be an automorphism, and let L be a line bundle with $L^r \otimes \sigma^* \Lambda^{-1} \cong \Lambda$. Send E to $L \otimes \sigma^*(E^\vee)$.

Note that the second type can only occur if $r|2d$. Also note that both operations send stable bundles to stable bundles, so they do define automorphisms of the moduli space $M(r, \Lambda)$.

We aim is to give a new proof of the following result of [KP] and [HR].

Theorem 1.1. *The automorphism group of $M = M(r, \Lambda)$ is given by the automorphisms described above.*

Theorem 1.1 was initially proved by Kouvidakis and Pantev [KP]. The proof of [KP] uses the Hitchin map defined on the moduli of Higgs bundles, giving a delicate argument in which one has to use the geometry of the fibers of the Hitchin map over spectral curves of three types: smooth curves, singular curves with a single node, and some singular curves with particular type of singularities.

In [HR], Hwang and Ramanan gave different proof of Theorem 1.1. The proof of [HR] is simpler in spirit (although it is a bit technical for the case $g = 4$). First, they determine geometrically the Hitchin discriminant (the locus of singular spectral curves), and then they go to the dual variety of this Hitchin discriminant, which they prove to be isomorphic to a locus of Hecke transforms. For this, one needs to use the theory of minimal rational curves in the moduli space.

We show that the proof of [HR] can be largely simplified. We prove Theorem 1.1 by reconstructing, from the Hitchin discriminant, the nilpotent cone bundles,

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which determine the automorphisms of a generic bundle directly. This avoids passing through the dual variety; it also avoids using both the Hecke transforms, and the minimal rational curves. Moreover, we cover all cases $g \geq 4$ with no special arguments for low genus.

It is worth pointing out that the argument provided here can be generalized to other moduli spaces, like the moduli space of symplectic bundles [BGM].

As a byproduct of our analysis, we also obtain the following Torelli type theorem for the moduli space $M(r, \Lambda)$. This is already well-known [MN, NR, NR2, KP, HR].

Theorem 1.2. *If $M_X(r, \Lambda) \cong M_{X'}(r', \Lambda')$ then $X \cong X'$ and $r = r'$.*

2. MODULI SPACE OF BUNDLES

Let X be a smooth complex projective curve of genus g , with $g \geq 3$. Fix $r \geq 2$, and also fix a line bundle Λ on X of degree d . A vector bundle $E \rightarrow X$ is called *stable* (respectively, *semistable*) if, for all proper subbundles $E' \subset E$ of positive rank,

$$\frac{\deg E'}{\operatorname{rk} E'} < \frac{\deg E}{\operatorname{rk} E} \quad (\text{respectively, } \frac{\deg E'}{\operatorname{rk} E'} \leq \frac{\deg E}{\operatorname{rk} E}).$$

Let $M(r, \Lambda)$ be the moduli space of stable bundles $E \rightarrow X$ of rank r and determinant $\det(E) \cong \Lambda$.

The following results are needed later. They appear in [HR] but we give a simpler proof.

Lemma 2.1. *Assume that $d \leq r(g-1)$ and $r \geq 2$.*

Then $H^0(E) = 0$ for a general stable bundle $E \in M(r, \Lambda)$.

Moreover, if $\mathcal{F} \subset \operatorname{Jac}^0 X$ is a κ -dimensional family of line bundles of degree zero and $d \leq r(g-1) - \kappa$, then for generic E ,

$$H^0(E \otimes L) = 0$$

for all $L \in \mathcal{F}$.

Proof. Recall that $\dim M(r, \Lambda) = (r^2 - 1)(g - 1)$. We want to show that the codimension of the subset

$$\{E \in M(r, \Lambda) \mid H^0(E) \neq 0\} \subset M(r, \Lambda)$$

is positive. Suppose that $E \in M(r, \Lambda)$ and $s \in H^0(E) \setminus \{0\}$. Then there is a short exact sequence

$$(2.1) \quad 0 \rightarrow L \xrightarrow{s} E \rightarrow Q \rightarrow 0,$$

where $L = \mathcal{O}_X(D)$ with D being the effective divisor defined by s . Write $\ell = \deg D \geq 0$. Let us compute the dimension of the space of extensions as in (2.1).

The line bundles L are parametrized by a family of dimension at most ℓ . The family of vector bundles Q is bounded, and the dimension of the parameter space is at most $((r-1)^2 - 1)(g-1)$. Finally, $\operatorname{Hom}(Q, L) = 0$, since otherwise there would be a non-constant non-zero endomorphism of E given by the composition $E \rightarrow Q \rightarrow L \hookrightarrow E$, which is not allowed by the stability condition of E . So

$$\dim \operatorname{Ext}^1(Q, L) = -\chi(L \otimes Q^*) = -\deg(L \otimes Q^*) + (r-1)(g-1) = d - r\ell + (r-1)(g-1),$$

since $\deg Q = d - \ell$. So the dimension of the family of E in (2.1) is

$$\begin{aligned} &\leq \ell + ((r-1)^2 - 1)(g-1) + d - r\ell + (r-1)(g-1) - 1 \\ &< (r^2 - 1)(g-1) - (r(g-1) - d). \end{aligned}$$

So if $d \leq r(g-1)$, then the family of E as in (2.1) is of positive codimension.

For the second part of the lemma, just note that the vector bundles E such that $H^0(E \otimes L') \neq 0$ for some $L' \in \mathcal{F}$ lie in a short exact sequence like (2.1) with $L = L' \otimes \mathcal{O}_X(D)$. The space of such bundles L is of dimension $\leq \ell + \kappa$. The result follows analogously as before. \square

For a bundle E , let $\text{End}_0 E$ denotes the sheaf of endomorphisms of trace zero. For any divisor D , define $\text{End}_0 E(D) := (\text{End}_0 E) \otimes \mathcal{O}_X(D)$.

Lemma 2.2. *Suppose that $g \geq 2\ell + 2$. Then $H^0(\text{End}_0 E(D)) = 0$ for a general stable bundle $E \in M(r, \Lambda)$ and any effective divisor D of degree ℓ .*

Proof. We will prove this by induction on r . The result is obvious for $r = 1$, so assume $r \geq 2$. Tensoring with a line bundle μ produces an isomorphism

$$M(r, \Lambda) \xrightarrow{\sim} M(r, \Lambda \otimes \mu^{\otimes r}), \quad E \mapsto E \otimes \mu.$$

Dualizing produces an isomorphism

$$M(r, \Lambda) \xrightarrow{\sim} M(r, \Lambda^*), \quad E \mapsto E^*.$$

In view of these isomorphisms, it suffices to prove the lemma for particular cases. So we can arrange that $0 \leq d \leq r/2$.

Using induction, a generic vector bundle $F \in M(r-1, \Lambda)$ satisfies the condition that $H^0(\text{End}_0 F(D)) = 0$, for any $D \in X^{(\ell)}$, where $X^{(\ell)} = \text{Sym}^\ell(X)$ is the set of effective divisors on X of degree ℓ . On the other hand, $H^0(F(D)) = 0$ and $H^0(F^*(D)) = 0$, for any $D \in X^{(\ell)}$. This is a consequence of Lemma 2.1, because

$$\pm d + (r-1)\ell \leq \frac{r}{2} + (r-1)\ell \leq (r-1)(g-1) - \ell,$$

which is equivalent to $(r-2)(\ell + \frac{1}{2}) + (r-1)(g-2\ell-2) \geq 0$.

Now let us construct a rank r vector bundle as follows. Take an extension

$$(2.2) \quad 0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow F \longrightarrow 0.$$

Let $f \in H^0(\text{End } E(D))$, i.e., $f : E \longrightarrow E(D)$. Composing f with the inclusion $\mathcal{O} \longrightarrow E$ and the projection $E(D) \longrightarrow F(D)$, we get a section of $F(D)$, which vanishes by construction. Therefore, f induces a homomorphism $f_2 : F \longrightarrow F(D)$. By induction hypothesis, $f_2 = s \text{ Id}$, where $s \in H^0(\mathcal{O}(D))$. The map $f' := f - s \text{ Id} \in H^0(\text{End } E(D))$ has $f'_2 = 0$, where f'_2 is constructed as above from f' . Consequently, it induces a map $f' : E \longrightarrow \mathcal{O}(D)$, i.e., a section of $E^*(D)$.

Let us see that we can choose an extension (2.2) so that $H^0(E^*(D)) = 0$. We have an exact sequence:

$$0 = H^0(F^*(D)) \rightarrow H^0(E^*(D)) \rightarrow H^0(\mathcal{O}(D)) \rightarrow H^1(F^*(D)) \rightarrow \dots$$

So we need an extension class $\xi \in H^1(F^*)$ so that the map $H^0(\mathcal{O}(D)) \xrightarrow{\xi} H^1(F^*(D))$ is injective for all $D \in X^{(\ell)}$. That is, ξ does not map to zero under $H^1(F^*) \rightarrow H^1(F^*(D))$. Looking at the exact sequence $0 \rightarrow V_D := F^*(D)|_D \rightarrow H^1(F^*) \rightarrow H^1(F^*(D)) \rightarrow 0$, we require $\xi \notin \bigcup_{D \in X^{(\ell)}} V_D$. This space is of dimension $\ell + (r-1)\ell = r\ell$, and $\dim H^1(F^*) = d + (r-1)(g-1)$. So the condition is

$$r\ell < d + (r-1)(g-1).$$

As $d \geq 0$, we need $g \geq \frac{r}{r-1}(\ell + 1)$. The minimum happens for $r = 2$, and it is $g \geq 2\ell + 2$.

Therefore, $f' \in H^0(E^*(D)) = 0$ and $f = s \text{Id}$. So we have constructed a vector bundle E such that $H^0(\text{End}_0 E(D)) = 0$ for any $D \in X^{(\ell)}$. Now take a family E_t , $t \in T$, of vector bundles parametrized by an (open) curve T , such that $E_0 = E$ and E_t is stable for generic t (this is possible because the moduli stack of vector bundles of fixed rank and determinant is irreducible and the stable locus is open in it [H]). As the condition that $H^0(\text{End}_0 E(D)) = 0$ is open, and $X^{(\ell)}$ is compact, we conclude that $H^0(\text{End}_0 E_t(D)) = 0$ for generic t . Hence $H^0(\text{End}_0 \tilde{E}(D)) = 0$ for generic stable bundle $\tilde{E} \in M(r, \Lambda)$. \square

Remark 2.3. Since $g \geq 4$, the condition on g in Lemma 2.2 is satisfied if $\ell = 1$.

3. HITCHIN DISCRIMINANT

Let us recall the definition of the Hitchin map (see [Hi, Section 5.10]). A Higgs bundle is a pair (E, θ) , where $\theta : E \rightarrow E \otimes K_X$. Denote $M = M(r, \Lambda)$ and let $\mathcal{M} = \mathcal{M}(r, \Lambda)$ be the moduli space of semistable Higgs bundles of rank r and determinant Λ . The cotangent bundle $T^*M \subset \mathcal{M}$ is an open subset.

The *Hitchin map*

$$H : \mathcal{M} \rightarrow W := H^0(X, K_X^2) \oplus \dots \oplus H^0(X, K_X^r),$$

is defined by $H(E, \theta) = (s_2(\theta), \dots, s_r(\theta))$, where $s_i(\theta) := \text{tr}(\wedge^i \theta)$. This restricts to

$$h : T^*M \rightarrow W \quad \text{and} \quad h_E : T_E^*M = H^0(X, \text{End}_0 E \otimes K_X) \rightarrow W.$$

For an element $s = (s_2, \dots, s_r) \in W$, the *spectral curve* X_s associated to it is the curve in the total space $\mathbb{V}(K_X)$ of K_X is defined by the equation

$$(3.1) \quad y^r + s_2(x)y^{r-1} + \dots + s_r(x) = 0,$$

where x is a coordinate for X , and y is the tautological coordinate dx along the fibers of the projection $\mathbb{V}(K_X) \rightarrow X$.

Consider

$$\mathcal{D} \subset W$$

the divisor consisting of characteristic polynomials with singular spectral curves.

Proposition 3.1. *The following statements hold:*

- (1) *For $w \in W - \mathcal{D}$, the fiber $h^{-1}(w)$ is an open subset of an abelian variety.*
- (2) *For generic $w \in \mathcal{D}$, the fiber $h^{-1}(w)$ is an open subset of the uniruled variety. The complement of this open subset is of codimension at least two.*

Proof. The map $H : \mathcal{M} \rightarrow W$ is proper. The inverse image $H^{-1}(w)$ is an abelian variety for $w \in W - \mathcal{D}$, by [Hi].

Now take $w \in W - \mathcal{D}$. If w is generic, then X_w has a unique singularity which is a node. Let Y be an integral curve whose only singularity is one simple node at a point y . Let $\pi_Y : \tilde{Y} \rightarrow Y$ be the normalization, and let x and z be the preimages of y . The compactified Jacobian $\bar{J}(Y)$, parametrizing torsionfree sheaves of rank 1 and degree 0 on Y , is birational to a \mathbb{P}^1 -fibration P over $J(\tilde{Y})$, whose fiber over $L \in J(\tilde{Y})$ is $\mathbb{P}(L_x \oplus L_z)$. The morphism $P \rightarrow \bar{J}(Y)$ is constructed as follows:

A point of P corresponds to a line bundle L on \tilde{Y} and a one dimensional quotient $q : L_x \oplus L_z \rightarrow \mathbb{C}$ (up to a scalar multiple). This is sent to the torsionfree sheaf L' on Y that fits in the short exact sequence

$$0 \rightarrow L' \rightarrow (\pi_Y)_* L \xrightarrow{q} \mathbb{C}_y \rightarrow 0.$$

For the proof, see [Bh, Theorem 4].

The complement

$$\mathcal{M} - T^*M$$

is of codimension at least three (in [Fa, Theorem II.6 (iii)] it is proved that the complement has codimension at least two under a weaker assumption, but if we assume $g \geq 3$, then the same proof gives that the codimension is at least three). Therefore, $(\mathcal{M} - T^*M) \cap \mathcal{D}$ is of codimension at least 2 in \mathcal{D} , so for generic $w \in \mathcal{D}$,

$$H^{-1}(w) - h^{-1}(w) \subset H^{-1}(w)$$

is of codimension at least 2. \square

Proposition 3.2. *The hypersurface $h^{-1}(\mathcal{D})$ is irreducible.*

Proof. Let $\mathcal{D}^* \subset \mathcal{D}$ be the complement of the subset consisting of all spectral curves with a single node. Then \mathcal{D}^* is closed so $\mathcal{D}^o = \mathcal{D} - \mathcal{D}^*$ is open. The fibers of h over points of \mathcal{D}^o are irreducible by Proposition 3.1. Hence $h^{-1}(\mathcal{D}^o)$ is irreducible. By Theorem II.5 of [Fa], the fibers of the Hitchin map $H : \mathcal{M} \rightarrow W$ are Lagrangian (hence their dimension is half the dimension of \mathcal{H}). So the fibers of H are equidimensional, and in particular, the codimension of $h^{-1}(\mathcal{D}^*)$ coincides with that of $\mathcal{D}^* \subset W$, which is at least two. Therefore $h^{-1}(\mathcal{D})$ is an irreducible divisor of T^*M . \square

The inverse image $h^{-1}(\mathcal{D})$ is called the *Hitchin discriminant*.

Theorem 3.3. *The Hitchin discriminant $h^{-1}(\mathcal{D})$ is the closure of the union of the (complete) rational curves in T^*M .*

Proof. Let $l \cong \mathbb{P}^1 \subset h^{-1}(\mathcal{D})$. Then $h(l) \subset W$. As it is a complete curve in an affine variety, it is a point. So l is included in a fiber. By Proposition 3.1, it cannot be contained in a fiber over $w \in W - \mathcal{D}$.

Now let $w \in \mathcal{D}^o$. Then Proposition 3.1 again shows that there is a family of \mathbb{P}^1 generically covering the fiber over w . Now using Proposition 3.2, we get that the closure is the entire $h^{-1}(\mathcal{D})$. \square

4. TORELLI THEOREM

This section is devoted to a *Torelli type theorem* for the moduli space $M = M(r, \Lambda)$.

Lemma 4.1. *The global algebraic functions $\Gamma(T^*M)$ produce a map*

$$\tilde{h} : T^*M \rightarrow \text{Spec}(\Gamma(T^*M)) \cong W \cong \mathbb{C}^N$$

which is the Hitchin map up to an automorphism of \mathbb{C}^N , where $N = \dim M$.

Moreover, consider the standard dilation action of \mathbb{C}^ on the fibers of T^*M . Then there is a unique \mathbb{C}^* -action “ \cdot ” on W such that \tilde{h} is \mathbb{C}^* -equivariant, meaning $\tilde{h}(E, \lambda\theta) = \lambda \cdot \tilde{h}(E, \theta)$.*

Proof. This holds for the Hitchin map H on the moduli of semistable Higgs bundles \mathcal{M} (cf. [Hi]). On the other hand, the general fiber of H is smooth, and the codimension of $T^*M \subset \mathcal{M}$ on these fibers is at least two (cf. [Fa, Theorem II.6 (i)], and note that T^*M is a subset of the moduli $\mathcal{M}^0 \subset \mathcal{M}$ of stable Higgs bundles). Therefore, we conclude that the lemma also holds for the restriction of the Hitchin map to the cotangent bundle T^*M .

The last assertion is clear. \square

We note that the \mathbb{C}^* -action on W allows to recover the space

$$W_r := H^0(K_X^r) \subset W$$

uniquely as the subset where the rate of decay is $|\lambda|^r$, which is the maximum possible.

Proposition 4.2. *The intersection $\mathcal{C} := \mathcal{D} \cap W_r \subset W_r = H^0(K_X^r) \subset W$ is irreducible. Moreover, $\mathbb{P}(\mathcal{C}) \subset \mathbb{P}(W_r)$ is the dual variety of $X \subset \mathbb{P}(W_r^*)$ for the embedding given by the linear series $|K_X^r|$.*

Proof. A spectral curve corresponding to a nonzero section $s_r \in W_r = H^0(K_X^r)$ has equation $y^r + s_r(x) = 0$, and this curve is singular at the points with coordinates $(x, 0)$ such that x is a zero of s_r of order at least two. We have $s_r \in \mathcal{C}$ if and only if there is some $x_0 \in X$ such that s_r vanishes at x_0 of order at least two. Therefore $s_r \in H^0(K_X^r(-2x_0)) \subset H^0(K_X^r)$. From this the second statement follows, taking into account that the linear system $|K_X^r|$ is very ample, so X is embedded. \square

Denote

$$\mathcal{C}_x = H^0(K_X^r(-2x)) \subset W_r.$$

Then $\mathcal{C} = \bigcup_{x \in X} \mathcal{C}_x$. There is a unique rational map $\mathcal{C} \rightarrow X$ which sends \mathcal{C}_x to x . This is uniquely determined, up to an automorphism of X , by the property that it is the only rational map with connected rational fibers, up to an automorphism of X .

Theorem 4.3 (Torelli). *Let X and X' be two smooth projective curves of genus $g \geq 3$, and consider two moduli spaces of stable vector bundles $M = M_X(r, \Lambda)$ and $M' = M_{X'}(r', \Lambda')$ on X and X' respectively. If these moduli spaces are isomorphic, then $X \cong X'$.*

Proof. Suppose $\Phi : M \rightarrow M'$ is an isomorphism. Then there is an isomorphism $d\Phi : T^*M \rightarrow T^*M'$. By Lemma 4.1, there is a commutative diagram

$$\begin{array}{ccc} T^*M & \xrightarrow{d\Phi} & T^*M' \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & W' \end{array}$$

for some isomorphism $f : W \rightarrow W'$. The \mathbb{C}^* -action by dilations on the fibers of T^*M and T^*M' induce \mathbb{C}^* -actions on W and W' , and f should be \mathbb{C}^* -equivariant (as $d\Phi$ is \mathbb{C}^* -equivariant). Therefore $f : W_r \rightarrow W'_r$, and $f|_{W_r}$ is linear.

We have seen in Proposition 3.1 that the Hitchin discriminant \mathcal{D} is an intrinsically defined subset, and therefore it is preserved by f . So f preserves $\mathcal{C} = \mathcal{D} \cap W_r$. This induces an isomorphism of the corresponding dual varieties, and hence by Proposition 4.2, an isomorphism $\sigma : X \rightarrow X'$ is obtained. \square

5. AUTOMORPHISMS OF THE MODULI SPACE

In this section we will compute the automorphism group of the moduli spaces of stable bundles on X . We assume that $g \geq 4$.

Recall that there is a decomposition

$$W = \bigoplus_{k=2}^r W_k = \bigoplus_{k=2}^r H^0(K_X^k).$$

Proposition 5.1. *Fix a generic stable bundle $E \in M = M(r, \Lambda)$, and consider the map*

$$h_r : H^0(\text{End}_0 E \otimes K_X) \longrightarrow W_r,$$

*given as composition of the Hitchin map on $H^0(\text{End}_0 E \otimes K_X) = T_E^*M \subset T^*M$, followed by the projection $W \longrightarrow W_r$. Then*

$$H^0(\text{End}_0 E \otimes K_X(-x_0)) = \{\psi \in H^0(\text{End}_0 E \otimes K_X) \mid h_r(\psi + \phi) \in \mathcal{H}_{x_0}, \forall \phi \in h_r^{-1}(\mathcal{H}_{x_0})\},$$

where $\mathcal{H}_x = H^0(K_X^r(-x)) \subset W_r = H^0(K_X^r)$, for $x \in X$.

Proof. First, note that the sequence

$$0 \longrightarrow H^0(\text{End}_0 E \otimes K_X(-x_0)) \longrightarrow H^0(\text{End}_0 E \otimes K_X) \longrightarrow \text{End}_0 E \otimes K_X|_{x_0} \longrightarrow 0$$

is exact, since $H^1(\text{End}_0 E \otimes K_X(-x_0)) = H^0(\text{End}_0 E(x_0))^* = 0$, for a general bundle (see Remark 2.3). So the map

$$H^0(\text{End}_0 E \otimes K_X) \longrightarrow \text{End}_0 E \otimes K_X|_{x_0},$$

given by $\phi \longmapsto \phi(x_0)$ is surjective.

Note that $h_r(\phi) = \det(\phi) \in W_r = H^0(K_X^r)$. So

$$h_r(\phi) \in \mathcal{H}_{x_0} \iff \det(\phi(x_0)) = 0.$$

The result now follows from this easy linear algebra fact: If V is a vector space and $A \in \text{End}_0 V$ satisfies the condition that $\det(A + C) = 0$ for any $C \in \text{End}_0 V$ with $\det(C) = 0$, then $A = 0$. \square

Proposition 5.1 allows us to construct the vector bundle

$$\mathcal{E} \longrightarrow X$$

whose fiber over $x \in X$ is $\mathcal{E}_x = H^0(\text{End}_0 E \otimes K_X(-x))$. This is a subbundle of the trivial vector bundle

$$H^0(\text{End}_0 E \otimes K_X) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow X,$$

and there is an exact sequence

$$(5.1) \quad 0 \longrightarrow \mathcal{E} \longrightarrow H^0(\text{End}_0 E \otimes K_X) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\pi} \text{End}_0 E \otimes K_X \longrightarrow 0.$$

(It is exact on the right by Remark 2.3.) So we recover the vector bundle

$$\text{End}_0 E \otimes K_X \longrightarrow X.$$

Let $x \in X$. We consider the nilpotent cone spaces

$$\mathcal{N}_{E,x} = \{A \in \text{End}_0 E \otimes K_X|_x \mid A^r = 0\} \subset \text{End}_0 E \otimes K_X|_x.$$

Consider the map

$$h_k : H^0(\text{End}_0 E \otimes K_X) \longrightarrow W_k$$

for $2 \leq k \leq r$, given as composition of the Hitchin map on $H^0(\text{End}_0 E \otimes K_X) = T_E^*M \subset T^*M$, followed by the projection $W \longrightarrow W_k$. The map h_k is defined by $(E, \phi) \longmapsto \text{tr}(\wedge^k \phi)$. Therefore,

$$(5.2) \quad \tilde{\mathcal{N}}_{E,x} := \bigcap_{k=2}^r h_k^{-1}(H^0(K_X^k(-x))) \subset H^0(\text{End}_0 E \otimes K_X)$$

is the preimage of the nilpotent cone under the surjective map $H^0(\text{End}_0 E \otimes K_X) \longrightarrow (\text{End}_0 E \otimes K_X)|_x$. Take its image to get $\mathcal{N}_{E,x}$. Letting x vary over X , we get the nilpotent cone bundle

$$\mathcal{N}_E \longrightarrow X$$

which sits as a sub-bundle

$$\mathcal{N}_E \subset \text{End}_0 E \otimes K_X.$$

So the moduli space $M = M(r, \Lambda)$ allows us to recover the nilpotent cone bundle. Let us see that we can also recover the flag bundle $Fl(E) \longrightarrow X$ of complete flags on the fibers of E .

Lemma 5.2. *Let V be a vector space, and define the nilpotent cone of V as:*

$$\mathcal{N} = \{A \in \text{End } V \mid A^r = 0\} = \{A \in \text{End}_0 V \mid A^r = 0\},$$

where $r = \dim V$. Then $\mathcal{N} \subset \text{End } V$ determines the flag variety $Fl(V)$ of complete flags in V .

Proof. This is a consequence of a theorem of Gerstenhaber [Ge]. Each complete flag in V determines a linear subspace $L \subset \mathcal{N} \subset \text{End } V$ consisting of nilpotent matrices respecting the flag. The dimension is

$$\dim L = \frac{1}{2}(r^2 - r) = \frac{1}{2} \dim \mathcal{N}.$$

Conversely, any linear subspace $L \subset \mathcal{N}$ of dimension $\frac{1}{2}(r^2 - r)$ is determined by a unique flag in this way. So the flag variety $Fl(V)$ is the space parametrizing these linear subspaces. \square

Now we are ready to prove the main result of the paper.

Theorem 5.3. *Let $M = M(r, \Lambda)$ be the moduli space of stable vector bundles. Let $\Phi : M \longrightarrow M$ be an automorphism. Then Φ is an automorphism of the type described in the introduction.*

Proof. As in the proof of Theorem 4.3, the automorphism Φ yields an isomorphism $\sigma : X \longrightarrow X$. Composing with an automorphism given by σ^{-1} , we may assume that $\sigma = \text{Id}$. Take a *general* bundle E , and let E' be its image by Φ . Then we have a diagram

$$\begin{array}{ccc} T_E^* M & \xrightarrow{F=d\Phi} & T_{E'}^* M \\ h \downarrow & & h \downarrow \\ W & \xrightarrow{f} & W \end{array}$$

where f is an automorphism which commutes with the \mathbb{C}^* -action.

We will show now that $f = \text{Id}$.

First note that $\text{Pic } M = \mathbb{Z}$ ([DN]). Therefore $\text{Pic } T^* M = \mathbb{Z}$. So the generic fiber of the Hitchin map has Picard group \mathbb{Z} . Let $s \in W$ be a generic point, and let $s' = f(s) \in W$. The isomorphism $d\Phi$ induces an isomorphism

$$F : h^{-1}(s) \longrightarrow h^{-1}(s').$$

Since the compactification $H^{-1}(s)$ of $h^{-1}(s)$ is an abelian variety, and $\text{codim}(H^{-1}(s) - h^{-1}(s)) \geq 2$, we conclude that F extends to an isomorphism

$$F : H^{-1}(s) \longrightarrow H^{-1}(s')$$

of polarized abelian varieties.

Let X_s be the spectral curve (3.1) associated to $s \in W$. Recall that $P_s := H^{-1}(s)$ is the Prym variety for the covering $X_s \rightarrow X$. Therefore there is an isomorphism of rational Hodge structures $H^1(X_s) \cong H^1(X) \oplus H^1(P_s)$. Moreover, the natural polarization of $H^1(X_s)$ is of the form $a\Theta_X + b\Theta_s$, where $a, b \in \mathbb{Q}_{>0}$, Θ_X is the natural polarisation of $H^1(X)$, and Θ_s is the unique polarisation of $H^1(P_s)$. We observe that this construction can be done over families. Therefore, a and b are constants. Since $(H^1(P_s), \Theta_s) \cong (H^1(P_{s'}), \Theta_{s'})$, we conclude that $H^1(X_s) \cong H^1(X_{s'})$ as polarized Hodge structures. By the usual Torelli theorem, $\tilde{f} : X_s \xrightarrow{\cong} X_{s'}$. The isomorphism $H^1(X_s) \cong H^1(X_{s'})$ preserves the factor $H^1(X)$. Therefore $\tilde{f} : X_s \rightarrow X_{s'}$ commutes with the natural projections $X_s \rightarrow X$ and $X_{s'} \rightarrow X$.

Recall that $X_s, X_{s'} \subset \mathbb{V}(K_X)$. The isomorphism sends each point $p \in X_s$ to a point $\tilde{f}(p) \in X_{s'}$. Both points are in the same fiber of $\mathbb{V}(K_X)$, so, if p is not in the zero section, the quotient $\tilde{f}(p)/p$ is a well-defined complex number. If we vary p , we obtain a rational function $\lambda(p)$ on X_s , such that \tilde{f} can be written as $\tilde{f}(p) = \lambda(p)p$. This function has poles at the intersection of X_s with the zero section of $\mathbb{V}(K_X) \rightarrow X$. So, it is of the form $\lambda = g/y$, where $y \in H^0(X_s, \pi^*K_X)$ is the tautological coordinate dx along the fibers of the projection $\mathbb{V}(K_X) \rightarrow X$ and g is a section of $H^0(X_s, \pi^*K_X)$.

Now

$$\begin{aligned} H^0(X_s, \pi^*K_X) &= H^0(X, (\pi_*\mathcal{O}_{X_s}) \otimes K_X) = H^0(X, \bigoplus_{j=0}^{r-1} K_X^{1-j}) = \\ &= H^0(X, \mathcal{O}_X) \oplus H^0(X, K_X). \end{aligned}$$

In this decomposition, the tautological section y is the identity on the first summand. Therefore, we can write $g = g_1y + g_2$, where $g_1 \in \mathbb{C}$ and $g_2 \in H^0(X, K_X)$. Therefore, the isomorphism \tilde{f} sends a point $p \in X_s$ with coordinate y to the point

$$\lambda(x)y = \left(g_1 + \frac{g_2}{y}\right)y = g_1y + g_2.$$

Therefore, the effect of $g_1 \in \mathbb{C}$ is to produce a dilation with constant factor, and the effect of g_2 is to produce a translation along the fiber. Note that g_2 is defined as a section on the curve X , so the translation is the same for all points of X_s over the same point of X . As we are requiring $s_1(x) = 0$ for the spectral curves, we must have $g_2 = 0$. Therefore, the isomorphism \tilde{f} is just a dilation by a (nonzero) constant factor.

Summarizing, we have proved that for general s ,

$$f(s) = \lambda_s s,$$

for some $\lambda_s \in \mathbb{C}^*$. This should then hold for all $s \in W$. Hence λ_s is constant, and f is multiplication by a constant λ . After scaling by λ^{-1} , we get that $f = \text{Id}$.

Now there is a vector bundle $\mathcal{E} \rightarrow X$, whose fiber over any $x \in X$ is the subspace $H^0(\text{End}_0 E \otimes K_X(-x)) \subset T_E^*M$. Analogously, there is a vector bundle $\mathcal{E}' \rightarrow X$ whose fiber over any x is the subspace $H^0(\text{End}_0 E' \otimes K_X(-x)) \subset T_{E'}^*M$. As $h \circ F = h$, Proposition 5.1 implies that $F : T_E^*M \rightarrow T_{E'}^*M$ gives an isomorphism $\mathcal{E} \rightarrow \mathcal{E}'$. Passing to the quotient bundle (5.1), we have an isomorphism of vector bundles

$$\text{End}_0 E \otimes K_X \rightarrow \text{End}_0 E' \otimes K_X.$$

The map F preserves the subspaces (5.2), again because $h \circ F = h$. Therefore there is an isomorphism

$$\begin{array}{ccc} \mathcal{N}_E & \longrightarrow & \mathcal{N}_{E'} \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

where \mathcal{N}_E and $\mathcal{N}_{E'}$ are the corresponding nilpotent cone bundles. By Lemma 5.2, we get an isomorphism

$$\begin{array}{ccc} Fl(E) & \longrightarrow & Fl(E') \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

of the corresponding flag variety bundles. Considering the global vertical fields, we have a Lie algebra bundle isomorphism

$$\begin{array}{ccc} \text{End}_0 E & \longrightarrow & \text{End}_0 E' \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

Using Lemma 5.4 below, it follows that $E' \cong E \otimes L$ or $E' \cong E^\vee \otimes L$, for some line bundle L . In the first case, it should be $L^r \cong \mathcal{O}_X$. In the second case, $L^r \cong \Lambda^2$ (and in particular $r|2d$).

As this holds for a generic E , it holds for all E . \square

Lemma 5.4. *Let E and E' be vector bundles of rank r such that $\text{End}_0 E$ and $\text{End}_0 E'$ are isomorphic as Lie algebra bundles. Then there is a line bundle L such that, either $E' \cong E \otimes L$, or $E' \cong E^\vee \otimes L$.*

Proof. Giving a vector bundle $\text{End}_0 E$ with its Lie algebra structure is equivalent to giving a principal $\text{Aut}(\mathfrak{sl}_r)$ -bundle $P_{\text{Aut}(\mathfrak{sl}_r)}$ which admits a reduction of structure group to a principal GL_r -bundle P_{GL_r} (the one corresponding to E).

We will study this reduction in two steps:

$$\text{GL}_r \twoheadrightarrow \text{PGL}_r = \text{Inn}(\mathfrak{sl}_r) = \text{Aut}(\mathfrak{sl}_r)^0 \hookrightarrow \text{Aut}(\mathfrak{sl}_r),$$

where $\text{Aut}(\mathfrak{sl}_r)^0$ is the connected component of the identity, which coincides with the inner automorphisms $\text{Inn}(\mathfrak{sl}_r)$. There is a short exact sequence of groups

$$1 \longrightarrow \text{Inn}(\mathfrak{sl}_r) \longrightarrow \text{Aut}(\mathfrak{sl}_r) \longrightarrow \text{Out}(\mathfrak{sl}_r) = \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Hence, the space of reductions of structure group of $P_{\text{Aut}(\mathfrak{sl}_r)}$ to PGL_r correspond to sections of the associated bundle $P_{\text{Aut}(\mathfrak{sl}_r)}(\text{Out}(\mathfrak{sl}_r))$. Since $\text{Out}(\mathfrak{sl}_r) = \mathbb{Z}/2\mathbb{Z}$, this associated bundle is a 2-to-1 cover of X , which is trivial (since we know that there are reductions), and therefore there are two reductions; these two reductions are related by an outer automorphism. Therefore, if $\mathbb{P} \rightarrow X$ is the bundle of projective spaces corresponding to one reduction, the other one is $\mathbb{P}^\vee \rightarrow X$, the dual projective bundle.

Now consider the short exact sequence of groups

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \text{GL}_r \longrightarrow \text{PGL}_r \longrightarrow 1.$$

The set of isomorphism classes of reductions of a PGL_r -bundle to GL_r is a torsor for the group $H^1(X, \mathcal{O}_X^*)$. Let E be a vector bundle corresponding to a reduction of $\mathbb{P} \rightarrow X$, i.e., $\mathbb{P}(E) \cong \mathbb{P}$. Then the reductions correspond to vector bundles of the form $E \otimes L$ for any line bundle L . On the other hand, E^\vee is a reduction of

$\mathbb{P}^\vee \longrightarrow X$. Therefore, if E is the vector bundle corresponding to a reduction of $P_{\text{Aut}(\mathfrak{sl}_r)}$, then all reductions are of the form either $E \otimes L$ or $E^\vee \otimes L$, where L is a line bundles. \square

Remark 5.5. Let $\overline{M}(r, \Lambda)$ be the moduli space of semistable vector bundles on X of rank r and determinant Λ . The automorphism group of $\overline{M}(r, \Lambda)$ coincides with that of $M(r, \Lambda)$. To see this, first note that since $g \geq 3$, the smooth locus of $\overline{M}(r, \Lambda)$ coincides with $M(r, \Lambda)$. Hence we have an injective homomorphism

$$\text{Aut}(\overline{M}(r, \Lambda)) \longrightarrow \text{Aut}(M(r, \Lambda)),$$

obtained by restricting maps. To prove that the above homomorphism is surjective, we note that for any line bundle ζ on $\overline{M}(r, \Lambda)$, the restriction homomorphism

$$(5.3) \quad H^0(\overline{M}(r, \Lambda), \zeta) \longrightarrow H^0(M(r, \Lambda), \zeta|_{M(r, \Lambda)})$$

is an isomorphism. Also, from the fact that the Picard group of $M(r, \Lambda)$ is \mathbb{Z} , it follows that the action of $\text{Aut}(M(r, \Lambda))$ on $\text{Pic}(M(r, \Lambda))$ is trivial. Now, take a very ample line bundle ζ on $\overline{M}(r, \Lambda)$. Since the homomorphism in (5.3) is an isomorphism, and any automorphism T of $M(r, \Lambda)$ fixes the restriction of ζ , we conclude that T acts on $H^0(\overline{M}(r, \Lambda), \zeta)$. The corresponding automorphism of $\mathbb{P}(H^0(M(r, \Lambda), \zeta))$ clearly fixes the image of $\overline{M}(r, \Lambda)$. Therefore, T extends to an automorphism of $\overline{M}(r, \Lambda)$.

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